The matrix of differentiation

Differentiation is a linear operation:

$$(f(x)+g(x))' = f'(x)+g'(x) \quad \text{and} \quad (cf(x))' = cf'(x).$$

Does it have a matrix?

In brief, the answer is yes. We need, however, to agree on the domain of the operation and decide on how to interpret functions as vectors. Consider an illustration.

Let $\mathcal{P}_2$ be the collection of all polynomials of degree at most 2, with real coefficients. Every polynomial $p(x) = a + bx + cx^2$ is completely determined by the vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ of its coefficients. The constant polynomial 1 corresponds to $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, x to $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and x^2 to $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Every polynomial in $\mathcal{P}_2$ is a linear combination of 1, x, and x^2, just as every vector in $\mathbb{R}^3$ is a linear combination of the orts $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. In this sense, $\mathcal{P}_2$ is very much like $\mathbb{R}^3$: addition and scaling work in the same way.

View $\mathcal{P}_2$ as the domain of the derivative operation. Differentiation maps 1 to 0, x to 1, and x^2 to $2x$. Using the above vector interpretation, we may write this correspondence as

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}.$$

Thus the derivative operation on $\mathcal{P}_2$ is represented by the matrix $D = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$.

The action $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} b \\ 2c \\ 0 \end{bmatrix}$ does agree with the formula $(a+bx+cx^2)' = b + 2cx$.

D is clearly not invertible: it lacks a pivot. It carries out a linear transformation of $\mathbb{R}^3$ which is neither one-to-one nor onto. Indeed, every vector of the form $\begin{bmatrix} * \\ 0 \\ 0 \end{bmatrix}$ is mapped to $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, and no vector with nonzero third component is in the range of D.

The range of D consists of vectors with zero third component. That's right, the derivative of a polynomial of degree at most 2 is of degree at most 1. Note that $D^2 = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ represents the second derivative on $\mathcal{P}_2$, and, of course, D^3 is the zero transformation.

Using the above framework, we may represent differentiation on $\mathcal{P}_3$ (polynomials of degree at most 3) by $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, differentiation on $\mathcal{P}_4$ (polynomials of degree at most 4) by $\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, and so on. The pattern is clear.